Journal of the Tennessee Academy of Science
Volume 49, Number 1, January, 1974

SEMIPRIME GOLDIE RINGS SATISFY THE INVARIANT (QUASI-) BASIS PROPERTY

James C. Pleasant
East Tennessee State University
Johnson City, Tennessee 37601

Abstract

A quasi-basis of a left R-module M is defined as a subset B of M such that every mapping B→F, where $F = {}_RF$ is (Levy) torsion-free injective, has a unique R-linear extension M→F. In an earlier paper the author gave an internal characterization of quasi-bases of modules over rings satisfying the common multiple property (CM) and established invariance of cardinality for quasi-bases of modules over integral domains satisfying (CM). In this paper, invariance of cardinality of quasi-bases is extended to modules over semiprime Goldie rings.

Introduction

A free (unitary left) R-module $M = {}_RM$ is said to be of rank n if it has a basis of cardinal n. The rank is not necessarily unique, but is unique if n is infinite (Cohn, 1966). The ring R is said to satisfy the invariant basis property (IBP) if the rank of every free left R-module is unique. Dieudonné (1942) proved that the following conditions are sufficient for R to satisfy IBP:

(1) R is commutative.

(2) R is (left) Noetherian.

(3) R is imbeddable in a division ring.

(4) R is imbeddable in a ring admitting a series of composition.

Any (left) Artinian ring with unity is (left) Noetherian (Lambek, 1966), hence satisfies IBP. We will make use of this fact in proving the theorem stated as the title of this paper. The interested reader will find further sufficient conditions in Cohn (1966).

Assumptions

In this paper all rings are assumed to have unity elements, and all modules are left unitary. The notation $M = {}_RM$ is used to mean M is a unitary left R-module. A quasi-basis of a left R-module M was defined as a subset B of M such that every mapping B→F (where $F = F_R$ (Levy) torsion-free injective) has a unique R-linear extension M→F (Pleasant, 1972). Over a ring R satisfying the left common multiple property, a quasi-basis of $M = {}_RM$ was shown to be characterized by the properties (a) B is linearly independent, and (b) the submodule <B> of M generated by B is "vital" in M (A submodule M′ of M is vital in M if for every $m \in M - M'$, therei s a regular element (i.e. non zero-divisor) $r \in R$ such that $rm \in M'$). Recently the author has learned of Megibben's use of the terminology quasi-basis of an abelian group G to mean a maximal independent subset of the set of elements of G having infinite order. It is easy to prove that this is a special case of the above definition.

A module $M = {}_RM$ with a quasi-basis B will be called quasi-free of rank n = card B. A ring R over which all quasi-free modules have invariant rank is said to satisfy the invariant quasi-basis property (IQBP). The purpose of this paper is to prove that semiprime Goldie rings satisfy IQBP. This strengthens Theorem 4 of our earlier paper (Pleasant, 1972) which stated that any integral domain satisfying the left common multiple property satisfies IQBP. It also generalizes Megibben's result establishing invariance of rank of an abelian group. Since any ring satisfying IQBP obviously satisfies IBP, our result yields as a corollary the apparently unknown fact that semiprime Goldie rings satisfy IBP.

Results

1. Some Preliminaries. The following lemmas and corollary are needed in the proof of our main result.

Lemma 1. If M′ is an essential submodule of $M = {}_RM$ and $m \in M - M'$, then the set $A = \{ a \in R \mid am \in M' \}$ is an essential left ideal of R.

Proof. It's easy to see that A is a left ideal. To show A is essential (i.e., A intersects each nonzero left ideal of R non-trivially), it suffices to prove that for every $b \in R - A$ there exists $b' \in R$ such that $0 \neq b'b \in A$. So let $b \in R - A$. Then bm is not in M-M′. Since M′ is essential in M, there exists $b' \in R$ such that $0 \neq b'(bm) = (b'b)m \in M'$. Thus $0 \neq b'b \in A$.

Goldie (1960) proved that for a semiprime Goldie ring R, every essential left ideal E of R contains a regular element. Combining this result with Lemma 1, we have the following corollary.

Corollary. Let R be a semiprime Goldie ring. Then any essential submodule M′ of a left-R-module ${}_RM$ is vital in ${}_RM$.

In Levy's (1963) torsion theory, an element m of a left R-module M is a torsion element if rm = 0 for some regular element r of R, and M is torsion-free if it

contains no nonzero torsion elements. If rM = M for each regular $r \in R$, then M is said to be divisible. A ring Q is called a left kuotient ring of a ring R if (1) R⊆Q, (2) every regular element of R has a two-sided inverse in Q, and (3) every element of Q has the form $r^{-1}a$ where $r, a \in R$ with r regular.

Lemma 2. If R has a left quotient ring Q and $M = {}_RM$ is (Levy) torsion-free divisible, then M is a left Q-module under the scalar multiplication: for $r^{-1}a \in Q$ and $m \in M$, $(r^{-1}a)m = m' \in M \rightleftarrows am = rm'$.

Proof. Divisibility of M assures that some $m' \in M$ satisfies this definition, while torsion-free-ness implies uniqueness of m′. The axioms for scalar multiplication are easily verified.

2. The Main Result. In the proof of the following theorem we use the fact that the quotient ring Q of a semiprime (left) Goldie ring R is (left) Artinian (Goldie, 1960.) Theorem 4.4, hence satisfies IBP.

Theorem. A semiprime Goldie ring R satisfies the invariant quasi-basis property.

Proof. Let $M = {}_RM$ have a quasi-basis $B = \{x_i\}_{i \in I}$ and let $F = {}_RF$ be the injective hull of M. Since M is essential in F, we see from the corollary to Lemma 1 that M is vital in F. Then since <B> is vital in M which is vital in F it easily follows that <B> is vital in F.

Let us consider first the special case in which $M = {}_RM$ is (Levy) torsion-free. Then $F = {}_RF$ is torsion-free ([8], Lemma 1) and divisible ([6], Theorem 3.1), hence by Lemma 2 becomes a module ${}_QF$ over the left quotient ring Q of R under the scalar multiplication $(r^{-1}a)x = y \Leftrightarrow ax = ry$ where $r^{-1}a \in Q$ and $x, y \in F$. We claim that $B = \{x_i\}_{i \in I}$ is a basis of ${}_QF$. To see that B generates ${}_QF$, consider $x \in F$. Since <B> is vital in ${}_RF$, there is a regular element r in R such that $rx \in$ <B>, say $rx = \sum_{i=1}^{n} a_i x_i$ where $a_i \in R$, $x_i \in B$. Thus we have $x = r^{-1}(rx) = \sum_{i=1}^{n} (r^{-1}a_i)x_i$, where $r^{-1}a_i \in Q$. To see that B is linearly independent over Q, suppose that $(r_1{}^{-1}a_1)x_1 + \ldots + (r_n{}^{-1}a_n)x_n = 0$ where $r_i{}^{-1}a_i \in Q$ and $x_i \in B$. Then there exist elements $\bar{a}_1, \ldots, \bar{a}_n$ and a regular element r of R such that $r^{-1}\bar{a}_i = r_i^{-1}a_i$, $i = 1, \ldots, n$ ([6], Lemma 1.3). Thus we have $(r^{-1}\bar{a}_1)x_1 + \ldots + (r^{-1}\bar{a}_n)x_n = r^{-1}(\bar{a}_1x_1 + \ldots + \bar{a}_nx_n) = 0$. It follows from the definition of scalar multiplication in ${}_QF$ that $\bar{a}_1x_1 + \ldots + \bar{a}_nx_n = 0$, so that $\bar{a}_1 = \ldots = \bar{a}_n = 0$ due to linear independence of B over R. Hence $r_i{}^{-1}a_i = 0$ for $i = 1, \ldots, n$ as desired. We have shown that B is a basis of ${}_QF$, a module over a (left) Artinian ring Q. Since Q satisfies IBP, we conclude that any two quasi-bases B and B′ of ${}_RM$, being bases of ${}_QF$, must have the same cardinality.

For the case where $M = {}_RM$ is not necessarily torsion-free, let T be the (Levy) torsion submodule of M. Then $M^* = M/T$ is torsion-free, hence satisfies IQBP. If $B_1 = \{x_i\}_{i \in I}$ and $B_2 = \{y_j\}_{j \in J}$ are quasi-bases of M, it is easy to show that $B^*_1 = \{x_i + T\}_{i \in I}$ and $B^*_2 = \{y_j + T\}_{j \in J}$ are quasi-bases of M^*, so that card B^*_1 = card B^*_2. But since card B_i = card B^*_i for i = 1,2, this yields card B_1 = card B_2.

Literature Cited

Cohn, P. M. 1966. Some remarks on the invariant basis property. Topology, 5: 215-218.

__________ Universal algebra. 1965. Harper and Row, New York. p. 82.

Dieudonné, J. 1942. Sur le nombre de dimensions d'un module. Comptus Rendus Acad. Sci. 215:563-565.

Goldie, A. W. 1960. Semiprime rings with maximum condition. Proc. Loudon Math Soc. 10: 201-220.

Lambek, J. 1966. Lectures on rings and modules. Blaisdell. p. 69.

Levy, L. 1963. Torsion-free and divisible modules over non-integral domains. Canad. J. Math. 15: 132-151.

Megibben, C. Introduction to Abelian Groups (Lecture notes). Vanderbilt University, Nashville.

Pleasant, J. C. 1972. Quasi-bases of modules. J. of Tennessee Acad. of Sci. 47: 122-124.

Journal of the Tennessee Academy of Science
Volume 49, Number 1, January, 1974

A TAXONOMIC AND ECOLOGICAL STUDY OF SOME ALGAE IN TWO PONDS IN SOUTH SHELBY COUNTY, TENNESSEE

Martha Powell Brown
Memphis State University
Memphis, Tennessee 38152

Abstract

Two ponds in south Shelby County were compared with respect to algal flora and ecology. Thirty-nine genera of algae were collected and identified and observations were made regarding the physical and chemical conditions in which they were found.