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LITERATURE CITED

Frankland, B. and P. F. Wareing. 1962. Changes in endogenous gibberellins in relation to chilling of dormant seeds. Nature 194:313-314.

Mayer, A. M. and A. Polijakoff-Mayber. 1963. *The Germination of Seeds*. The MacMillian Co., N.Y. 236 pp.

McVaugh, Rogers. 1936. Studies on the taxonomy and distribution of the Eastern North American species of *Lobelia*. Rhodora 38:241-263, 276-298, 305-329, 346-362.

Quarterman, Elsie. 1950. Major plant communities of Tennessee cedar glades. Ecology 31:234-254.

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POINTS OF DIVERGENCE FOR THE ITERATION OF MEROMORPHIC FUNCTIONS

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INTRODUCTION

Let F denote a meromorphic function, and consider the fixed-point problem

(1) $F(z) = z.$

If for arbitrary complex z we construct the sequence $\{z_n\}$ defined by

(2) $z_0 = z, \quad z_n = F^n(z) \qquad (n = 1, 2, \ldots)$

where F^n denotes the n-th iterate of F, then the following facts are well known:

(a) If the sequence (2) converges to a finite point ζ, then ζ must be a solution of (1).

(b) Under a suitable hypothesis on the form of the analytic expansion of F about one of its fixed points ζ, the sequence (2) converges to ζ provided the starting value $z = z_0$ is taken sufficiently near ζ.

(c) If F has the form

(3) $F(z) = z - f(z)g(z),$

and if ζ is a finite fixed point of F, then ζ is a root of the equation

(4) $f(z) = 0$

provided $g(\zeta) \neq 0.$

Remarks (a) and (c) are obvious, and (b) is formulated as Theorem 1 of section 2. We refer the reader to Hochstrasser [4] for a more complete discussion of these matters.

It has been noted in various connections that an unfortunate choice of a starting value in (2) may produce a divergent sequence. A familiar example is Newton's method $(g(z) = 1/f'(z)$ in equation (3)) and the encounter of a singularity of F $(f'(z_n) = 0$ for some $n)$. In fact, the convergence of Newton's method on the real axis and in the complex plane has received careful attention in recent years (see, for example, the papers of Corn [3] and Barna [1]).

In the present paper we shall be concerned with the sets $C(\zeta)$ of starting values for the finite fixed points ζ of F and in particular with the set D of points of divergence, that is points z for which the sequence (2) diverges or ceases to be defined at some stage. Under combinations of the assumptions (I) and (II) of section 2, we shall prove that

(i) the convergence of the sequence $\{F^n(z)\}$ is uniform in any compace subset of $C(\zeta)$;

(ii) an isolated point of D must be a pole of F^n for some n,

(iii) a pole of F^n is an isolated point of D if and only if the set $D - \{\infty\}$ is bounded.

Since we shall see that D is closed in the topology of the extended complex plane, we note that (ii) and (iii) furnish a sufficient condition that D be a perfect set.

FUNDAMENTAL ASSUMPTIONS; THE CLASSICAL CONVERGENCE THEOREM.

Throughout the discussion we shall assume that the function F is mermorphic. We shall also have occasion to assume that F has one or both of the following properties:

(I) F has a (non-removable) singularity at infinity.

(II) About each of its finite fixed points ζ, F has an expansion of the form

(5)

$$F(z) = \zeta + \sum_{n=1}^{\infty} a_n (z - \zeta)^n \quad \text{with} \quad |a_1| < 1.$$

Ritt [5] gave the name *point of attraction* to any point ζ about which an expansion of the form (5) is valid. The classical local convergence theorem is

THEOREM 1. *If ζ is a point of attraction of F, there exists an $\varepsilon > 0$ such that the sequence (2) converges to ζ for every z satisfying $|z - \zeta| < \varepsilon$.*

For proof see [4].

We remark that if ε of Theorem 1 is sufficiently small, the inequality $|z_n - \zeta| < \varepsilon$ implies $|z_{n+p} - \zeta| < \varepsilon$ $(p = 1, 2, \ldots)$. This implication may be derived from the estimates used in the proof of the theorem and will be needed in the discussion of the uniformity of the convergence.

A second necessary remark is that if F has property (II), then each $C(\zeta)$ is an open set. For by Theorem 1, $C(\zeta)$ contains an open disk N which contains ζ. We may write

$$(6) \qquad C(\zeta) = \bigcup_{n=1}^{\infty} F^{-n}(N)$$

where F^{-n} denotes the inverse of F^n. Since F^n is continuous on $C(\zeta)$ for each n, $F^{-n}(N)$ is open; hence the union (6) is open.

It follows that if F has property (II), D is a closed set.

UNIFORMITY OF CONVERGENCE

THEOREM 2. *If F has property* (I) *and if ζ is a point of attraction of F, then the convergence of $\{F_n(z)\}$ is uniform in every compact subset of $C(\zeta)$.*

Proof. We first note that the assumption of property (I) implies that $\infty \in D$; hence for subsets of $C(\zeta)$ the term compact is equivalent to closed and bounded. Let us assume, therefore, that the convergence of $\{F^n(z)\}$ is not uniform on some closed and bounded subset S of $C(\zeta)$. Then there exists an $\varepsilon > 0$ with the property that the subsets S_n of S, defined by the inequality

$$|F^n(z) - \zeta| \geqq \varepsilon$$

are nonempty for infinitely many n. Without loss of generality we may suppose that ε is small enough to insure that

(7) $|z - \zeta| < \varepsilon$ implies $|F(z) - \zeta| < \varepsilon$

(see the first remark following Theorem 1).

Because each $S_n \subset C(\zeta)$, there exists an increasing sequence $\{n_i\}$ of suffixes with the property that

$$S_{n_i} \neq \emptyset \quad \text{and} \quad S_{n_i} \supset S_{n_{i+1}} \qquad (i = 1, 2, \ldots)$$

where the inclusion is proper. If we choose

$$w_i \in S_{n_i} - S_{n_{i+1}} \qquad (i = 1, 2, \ldots)$$

the set of points $\{w_i\,;\, i = 1, 2, \ldots\}$ is bounded and infinite and thus has a limit point $w \in S$.

Since $w \in C(\zeta)$, there exists an m such that $|F^m(w) - \zeta| < \varepsilon$. By continuity of F^m at ω, each z in an entire neighborhood N of ω must satisfy $|F^m(z) - \zeta| < \varepsilon$, and hence, by (7), we must have

(8) $|F^{m+p}(z) - \zeta| < \varepsilon \qquad (p = 0, 1, \ldots)$

for every $z \in N$.

But N contains w_i for infinitely many i. Choose i such that $w_i \in N$ and $n_i > m$. For $p = n_i - m$ we must have

$$|F^{m+p}(w_i) - \zeta| = |F^{n_i}(w_i) - \zeta| \geqq \varepsilon;$$

the last inequality is by definition of S_{n_i}. This is a contradiction of (8).

ISOLATED POINTS OF D

We begin our discussion of isolated points of D by showing that the points of any two distinct convergence sets are separated from each other by points of D.

THEOREM 3. *Let F have property* (II), *and let ζ_0 and ζ_1 be distinct finite fixed points of F. Suppose $z_i \in C(\zeta_i)$, $i = 0, 1$, and let γ be any continuous curve with endpoints z_0 and z_1. Then at least one point of D lies on γ.*

Proof. Represent γ parametrically by

$z = \Phi(t)$, $0 \leqq t \leqq 1$, where $z_0 = \Phi(0)$, $z_1 = \Phi(1)$.

Let $\tau = \sup\{t \in [0, 1]\,;\, \Phi(t) \in C(\zeta_0)\}$. The point $\Phi(\tau)$ is easily seen to belong to D since each $C(\zeta)$ is an open set.

We have now laid the groundwork for

THEOREM 4. *Let F have properties* (I) *and* (II). *If α is a finite isolated point of D, then α is a pole of F^n for some n.*

Proof. If α is an isolated point of D, one easily shows by applying Theorem 3 that there exists a deleted neighborhood N' of α which is entirely contained in $C(\zeta)$ but which excludes α. Let 2δ be the radius of N', and let γ denote the circle of radius δ with center α. We shall denote by M the disk of radius $\varepsilon > 0$ with center ζ, and we shall suppose that ε has been chosen sufficiently small that

$$F(M) \subset M \subset C(\zeta).$$

We now prove the following: If for every n $F^n(\alpha)$ is defined and finite (and hence F^n is analytic at each point of $N' \cup \{\alpha\}$), then the sequence $\{F^n(\alpha)\}$ converges, contrary to the hypothesis of the theorem. It suffices to find one value of n such that $|F^n(\alpha) - \zeta| < \varepsilon$. Since γ is a compact subset of $C(\zeta)$, choose n such that $|F^n(z) - \zeta| < \varepsilon$ for each $z \in \gamma$. Using Cauchy's integral formula, we obtain

$$|F^n(\alpha) - \zeta| = \left| \frac{1}{2\pi i} \int_\gamma \frac{F^n(z)\,dz}{z - \alpha} - \frac{1}{2\pi i} \int_\gamma \frac{\zeta\,dz}{z - \alpha} \right|$$

$$\leq \frac{1}{2\pi} \int_\gamma \frac{|F^n(z) - \zeta|}{|z - \alpha|} |dz| < \frac{1}{2\pi} \frac{\varepsilon}{\delta} 2\pi\delta = \varepsilon.$$

It follows by contradiction that α must be a singularity of F^n for some n. If α were an essential singularity of F^n, then α would be a pole of F^{n-1} since F is meromorphic. This completes the proof.

We shall need the following lemma from the theory of functions (see Dienes [2, p. 246]) for the proof of Theorem 5.

LEMMA. *If G is analytic at the finite point α, and if G(α) = 0, then there exists a function H, which is analytic at 0, and an integer p ≧ 1, such that*

$$H(0) = \alpha \tag{9}$$

and the equation

$$u = G(z)$$

is solved by the relation

$$z = H(\sqrt[p]{u})$$

for all u in a sufficiently small neighborhood of 0.

THEOREM 5. *A finite pole α of F^n is isolated from D if and only if $D - \{\infty\}$ is bounded.*

Proof. Write $G(z) = 1/F^n(z)$. Then G satisfies the hypotheses of the lemma. If $D - \{\infty\}$ is unbounded, select a sequence $\{\beta_i\}$, $\beta_i \in D - \{\infty\}$, with $\lim_{i\to\infty} \beta_i = \infty$. If $u_i = 1/\beta_i$ and $z_i = H(\sqrt[p]{u_i})$, then the z_i must cluster about α. But $F^n(z_i) = 1/G(z_i) = 1/u_i = \beta_i$, which implies $z_i \in D$. Hence α cannot be isolated from D.

On the other hand, points of D which cluster about α have images under F^n which cluster about ∞.

EXAMPLES

(i) $F(z) = 1/z$. Assumptions (I) and (II) are both violated. The only fixed points are $\zeta = \pm 1$. The sets $C(1)$ and $C(-1)$ consist of the single points 1 and −1 respectively, and every other point of the extended plane belongs to D. Here D is open and $C(1)$ and $C(-1)$ are closed.

(ii) $F(z) = z^2$. The points 0 and 1 are the only finite fixed points. The point 0 is a point of attraction, while 1 is not. Since $F^n(z) = z^k$, where $k = 2^n$, we have

$$\lim_{n\to\infty} F^n(z) = \begin{cases} 0 \text{ if } |z| < 1 \\ \infty \text{ if } |z| > 1. \end{cases}$$

Hence the interior of the unit disk is contained in $C(0)$, and the exterior is contained in D. If $n > 1$, the fixed points of the function F^n are 0, 1, and the non-real $(2^n - 1)$-th roots of 1. Each such root ω has the property that

$$F(\omega) \neq \omega$$

and

$$\omega = F^n(\omega) = F^{2n}(\omega) = \ldots$$

Hence $\omega \in D$. Such points of D have been called *cyclic points* (see Gorn's introduction [3]). These points are obviously dense in the unit circle.

Every solution of the equations

$$z^k - 1 = 0, \text{ where } k = 2^n \qquad (n = 1, 2, \ldots)$$

is an element of $C(1)$. These points are also dense in the unit circle. It follows that the set $C(1)$ is not closed, for its closure contains cyclic points; neither is it open, for every neighborhood of each of its points contains points of $C(0)$. A similar remark shows that D is neither open nor closed.

We finally note that the conclusion of Theorem 3 does not hold for this example: the unit interval connects the two fixed points and contains no point of D.

(iii) $F(z) = (2/3)z + 1/(3z^2)$. This is the Newton transform of the polynomial $f(z) = z^3 - 1$. The finite fixed points of F are the three cube roots of 1; each of these is a point of attraction, so that F has property (II). Moreover, F has property (I) since ∞ is a pole of F.

We note that $0 \in D$. It is not difficult to show that if $\alpha \leqq 0$, the equation

$$F(z) = \alpha$$

has one real root $\alpha' < \alpha$. If the real sequence $\{\alpha_n\}$ is defined by the formulas

$$F(\alpha_1) = 0, \quad F(\alpha_{n+1}) = \alpha_n,$$

one may prove that $\lim_{n\to\infty} \alpha_n = \infty$. Hence $D - \{\infty\}$ is unbounded, and it follows from Theorems 4 and 5 that the set D is perfect.

LITERATURE CITED

1. Barna, B. 1961. Über die Divergenzpunkte des Newtonsche Verfahrens zur Bestimmung von Wurzeln algebraischer Gleichungen III. *Publ. Math. Debrecen* 8:193-207.
2. Dienes, P. 1957. The Taylor Series. Dover, New York.
3. Gorn, S. 1954. Maximal convergence intervals and a Gibbs type phenomenon for Newton's approximation procedure. *Ann. of Math.* 59:463-476.
4. Hochstrasser, U. 1962. Numerical methods for finding solutions of nonlinear equations, from A Survey of Numerical Analysis, J. Todd, editor. McGraw-Hill, New York.
5. Ritt, J. F. 1920. On the iteration of rational functions. *Trans. Amer. Math. Soc.* 23:348-356.